A time-domain Hilbert transformer, also referred to as a quadrature filter or a wideband $\pi$ phase shifter, is a device that calculates the Hilbert transform of an arbitrary input temporal signal [1]. In the electronic domain, Hilbert transformers are fundamental devices for numerous applications, e.g., in communications, computing, information processing, signal analysis and measurement [1]. A similar range of applications could be expected for a photonic Hilbert transformer (PHT), with the essential difference that such a device would enable processing signals directly in the all-optical domain and at speeds (operation bandwidths) well beyond the reach of electronic technologies. PHTs based on linear discrete-time photonic filtering schemes, i.e., multitap fiber-optics transversal filters [2–4] and sampled fiber Bragg gratings (FBGs) [5], have been recently demonstrated. The demonstrated multitap PHTs [2–4] are specifically adapted for processing microwave signals, and they have been successfully employed for high-frequency single-sideband (SSB) microwave modulation [2,3] and instantaneous microwave frequency measurement [4]. Besides its intrinsic implementation complexity, it has been estimated that the discrete-time PHT solution is limited to operation bandwidths <40 GHz [3].

In this Letter, we propose and numerically demonstrate a new and simple PHT design based on a single all-fiber device, namely, a phase-shifted uniform-period FBG with a suitable amplitude-only apodization profile operating in reflection (linear regime). The proposed concept is based on an analog design (in contrast to previous discrete-time schemes) and allows implementing all-fiber PHTs capable of processing arbitrary optical waveforms with bandwidths up to a few hundreds of gigahertz.

Let us assume an optical signal defined as (analytic representation) $x(t)e^{j2\pi f_0 t}$, where $x(t)$ is the signal’s temporal complex envelope ($t$ is the time variable) and $f_0$ is the optical carrier frequency. This signal is launched at the input of an ideal PHT. The output from the PHT is an optical waveform $y(t)e^{j2\pi f_0 t}$ with a complex temporal envelope given by the Hilbert transform of the input temporal envelope [1],

$$y(t) \propto \text{P.V.} \left[ \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \right],$$

where the symbol P.V. indicates principal value.

The frequency response of an ideal PHT is plotted with a dotted curve in Fig. 1(a), and the physically realizable PHT proposed in this Letter (solid curves). The frequency response of an ideal PHT [plotted with a dotted curve in Fig. 1(a)] exhibits a flat amplitude for all the frequency variables ($\pm f_0$) and a $\pi$-phase shift at the central carrier frequency $f_0$. The temporal impulse response of the ideal PHT is plotted in Fig. 1(b), dotted curve (temporally shifted to facilitate a direct comparison with the impulse response of a physically realizable PHT). The frequency response of an ideal

![Fig. 1. Schematic showing the (a) baseband spectral transfer function and (b) envelope of the temporal impulse response of an ideal PHT (dotted curves) and the physically realizable PHT proposed in this Letter (solid curves).](image-url)
PHT is not realizable in practice, because (i) it exhibits an infinite frequency bandwidth, (ii) its associated impulse response has an infinite duration and is non-causal \( h(t) \neq 0 \) for \( t < 0 \), and (iii) the corresponding temporal impulse response tends to infinity at \( t = 0 \). To find a physically realizable PHT one can limit the frequency response bandwidth \( (\Delta f) \) and the temporal impulse response duration \( (\Delta T) \) of the filter. A time-domain shift in the filter’s impulse response, e.g., by \( T_c = \Delta T/2 \), is also needed to ensure causality. The spectral and temporal impulse responses of the resulting realizable PHT are plotted in Figs. 1(a) and 1(b) (solid curves), respectively. The temporal impulse response of the physically realizable PHT can be easily calculated as the inverse Fourier transform \( (\mathcal{F}^{-1}) \) of the corresponding bandwidth-limited frequency spectral response, \( H(f) \), followed by the above mentioned time-window truncation and temporal shifting processes

\[
\begin{align*}
    h(t) & \propto (\mathcal{F}^{-1}\{H(f)\} \times \Pi(t/\Delta T)) \ast \delta(t - T_c) \\
    & \approx \left[ \frac{\left( f + \Delta f/4 \right)}{\Delta f/2} \right] \frac{\left( f - \Delta f/4 \right)}{\Delta f/2} \times \Pi(t/\Delta T) \ast \delta(t - T_c) \\
    & \approx \frac{\sin^2(\pi(t - T_c)/\Delta f/2)}{(t - T_c)/\Delta f} \Pi(t/\Delta T),
\end{align*}
\]

where * indicates convolution and the function \( \Pi(l) \) is the squared pulse with a total duration of \( \Delta l = 1 \) centered at \( l = 0 \). As shown in Fig. 1(b), the obtained temporal impulse response is a causal time-limited and amplitude-limited version of the ideal PHT temporal impulse response.

The required temporal impulse response for a physically realizable PHT can be implemented using a simple weak-coupling uniform-period FBG with a proper grating apodization profile. The reflection temporal impulse response of a weak-coupling FBG, \( h(t) \), is approximately proportional to the grating apodization profile, i.e., variation of the refractive-index peak modulation \( \Delta n(z) \) as a function of the grating length \( z \), after the suitable space-to-time scaling [6]. Thus, to obtain the temporal impulse response given by Eq. (2) along a finite-time interval \( 0 \leq t \leq \Delta T \), the grating apodization profile should vary as follows:

\[
\Delta n(z) \approx \frac{\sin^2(\pi n_{av} \Delta f/(2z_c))}{(z - z_c)},
\]

over the fiber length \( 0 \leq z \leq L \), with \( L = c \Delta T/(2n_{av}) \)
\( = c T_c/n_{av} \), and \( z_c = L/2 \), where \( n_{av} \) is the average refractive-index peak value of the uniform-period FBG, \( c \) is the speed of light in vacuum, and \( L \) is the total grating length. The resulting grating profile is very simple, requiring a single \( \pi \)-phase shift along its length. A longer temporal impulse response (longer \( \Delta T \), corresponding to a smaller \( \Delta f \) ) can be achieved using longer gratings. On the other hand, the zero-to-zero width of the sidelobes, \( \Delta z \), in the apodization profile is inversely related to the full operation spectral bandwidth of the PHT, i.e., \( \Delta z = c/(n_{av} \Delta f) \) [see Eq. (3) and also the illustration of this relationship in Fig. 1]. Therefore a broader processing bandwidth can be obtained using narrower apodization sidelobes. Current FBG fabrication technologies allow writing grating apodization profiles with spatial resolutions as short as \( \approx 100 \mu m \) and with total grating lengths easily in the tens-of-centimeters range. Hence, we estimate that PHTs with operation bandwidths up to a few hundreds of gigahertz and capable of processing optical pulses extending over nanosecond time windows could be fabricated with present FBG writing technologies.

An FBG-based PHT was designed and numerically tested based on the ideas introduced above. We targeted a full operation bandwidth of \( \Delta f = 150 \) GHz and a temporal impulse response duration of \( \Delta T \approx 200 \) ps (corresponding to a lower bandwidth limit of \( \Delta f = 5 \) GHz). The effective refractive index of the mode in the unperturbed single-mode fiber (SMF) was assumed to be \( n_{eff} = 1.452 \), and the grating period was fixed to \( \Lambda = 534.18 \) nm, which corresponds to a Bragg frequency \( (f_b) \) of 193 THz. To obtain the desired operation time window \( (\Delta T \approx 200 \) ps), the total FBG length was fixed to \( L = 2 \) cm. Even though our design is based on the use of first-order Born approximation (weak-coupling FBGs), our simulations showed that the targeted frequency spectral response can still be achieved with a sufficient accuracy using a mid-strength FBG. In the design reported here, the grating peak-to-peak index modulation was set to \( \Delta n_{pp} = 7.2 \times 10^{-4} \) corresponding to a maximum reflectivity of \( \approx 45 \)%, so as to optimize the device’s energetic efficiency. The FBG was modeled and simulated using the coupled-mode theory and transfer-matrix techniques [7]. To achieve the targeted operation bandwidth of \( \Delta f = 150 \) GHz, the sidelobes’ zero-to-zero width in the apodization profile was fixed to \( \Delta z = c/(n_{av} \Delta f) = 1.37 \) mm. The designed apodization profile is shown in the inset of Fig. 2(b). As anticipated, the resulting grating apodization profile requires a single \( \pi \)-phase shift along the device’s length. Figure 2(a) presents the reflectivity of the designed PHT. The insets of Fig. 2(a) show the phase profile of the grating’s reflection spectral response (for illustration purposes, the linear phase change, associated with the device’s average group-delay, is not represented) and the corresponding reflection amplitude spectrum in decibels around the central frequency. The spectral reflection response of the grating exhibited nearly a flat amplitude (maximum amplitude variation \(< 2 \) dB) over the \( \approx 150 \) GHz total operation bandwidth with a dip of \(< 50 \) dB and the required \( \pi \)-phase shift at the central frequency. Figure 2(b) (solid curve) shows the amplitude of the FBG reflection temporal impulse response, calculated as the inverse Fourier transform of the FBG reflection complex spectral response. For comparison, the targeted temporal impulse response is also shown in Fig. 2(b).
an ideal PHT, i.e., 50 GHz cosine function, is also the designed FBG. For comparison, the output from envelope of the waveform obtained after reflection from 3(b) (solid curve) shows the normalized temporal en-
signal of 50 GHz; see Fig. 3(a) (dashed curve). Figure the input temporal envelope was a microwave sine
signal by the FBG's average group-delay
the output signal is delayed with respect to the input
center of the corresponding waveform. In reality,
put and output signals has been fixed to coincide with
tration purposes, the time reference
form of the input temporal envelope (circles).
reflected optical signal from the FBG-based PHT (solid
input temporal envelope (dashed curve), the envelope of the

time change of the reflection FBG spectral response and corre-
responding amplitude in decibels around the central fre-
cquency. (b) Envelope amplitude of the reflection temporal
waveform); (a) 50 GHZ sinusoid; (b) first-order derivative of
Fig. 3. (Color online) Results from numerical simulations
for the designed PHT assuming an input signal (complex
envelope); (a) 50 GHZ sinusoid; (b) first-order derivative of
a Gaussian pulse with FWHM=15 ps. Each plot shows the
input temporal envelope (dashed curve), the envelope of the
reflected optical signal from the FBG-based PHT (solid
curve), and the numerically calculated ideal Hilbert trans-
form of the input temporal envelope (circles).

(circles), showing an excellent agreement with the
designed FBG's temporal impulse response in the
200 ps operation time window.

The operation of the designed PHT was confirmed
by simulating the temporal response of this grating
to various input optical waveforms, all of them spec-
trally centered at $f_0=193$ THz. In the first example,
the input temporal envelope was a microwave sine
signal of 50 GHz; see Fig. 3(a) (dashed curve). Figure
3(b) (solid curve) shows the normalized temporal en-
velope of the waveform obtained after reflection from
the designed FBG. For comparison, the output from
an ideal PHT, i.e., 50 GHz cosine function, is also
plotted with circles in Fig. 3(a). Notice that, for illus-
tration purposes, the time reference ($t=0$) of the
input and output signals has been fixed to coincide with
the center of the corresponding waveform. In reality,
the output signal is delayed with respect to the input
signal by the FBG's average group-delay ($\approx 97$ ps).

Notice that the designed PHT does not impose any
limitation on the input and output time duration for
single-frequency signals as long as the frequency is
within the operation bandwidth. In the second ex-
ample, the input waveform was a first-order Hermite–Gaussian pulse (typical test signal with a
well-known Hilbert transform [1]), defined as the
first-order derivative of a transform-limited 15 ps
(FWHM) Gaussian optical pulse; see Fig. 3(b) (dashed curve). Figure 3(b) (solid curve) shows the
normalized temporal envelope of the waveform
obtained after reflection from the designed PHT. In all
of the cases, there was an excellent agreement be-
tween the simulated output waveform and the ideal
(numerically calculated) Hilbert transform of the in-
put waveform. Since the designed device is basically
an all-pass filter, the energetic efficiency (ratio be-
tween the output signal energy over the input signal
energy) is always high with the only restriction being
that the FBG designed here exhibits a 45% peak re-
flectivity. Moreover, the processing accuracy (normal-
ized cross correlation between the simulated and the
ideal outputs) of the device is also nominally high as
long as the input pulse frequency spectrum is within
the operation bandwidth of the device. To give a ref-
ference, for the simulated case plotted in Fig. 3(b), an
energetic efficiency of 34.36% and a processing accu-
rac of 99.71% were achieved. To examine the re-
quired accuracy for the $\pi$-phase shift in the grating
apodization profile we have numerically estimated
that to keep the device processing accuracy higher
than 99.5% a relatively large deviation of $\pm 0.4$ rad
can be accepted over the nominal $\pi$-phase shift.

In summary, we have proposed and numerically
demonstrated an FBG-based design for an all-fiber
PHT suitable for processing both broadband micro-
wave signals (modulated on an optical carrier) and
ultrafast optical waveforms. The proposed design is
based on a single all-fiber device, namely, a specially
apodized uniform-period FBG incorporating a single
$\pi$-phase shift. This approach would allow implement-
ing all-fiber PHTs capable of calculating the Hilbert
transform of arbitrary optical temporal waveforms
(complex envelopes) with bandwidths up to a few
hundreds of gigahertz using readily feasible FBGs.

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